Math 256 Midterm #1 *February 7, 2013*

Name: Student number:

Open Book Section

Prob. 1. (16 pts.)

Consider the non-homogeneous ODE

$$d^2y/dt^2 - y = 4t^2$$

(i) Find the two fundamental solutions to the homogeneous equation. $y'' - y = 0$

Characteristic eq. $r^2 - 1 = 0$ $r = \pm 1$

$y_1 = e^t, \quad y_2 = e^{-t}$

(ii) Verify that they are linearly independent.

$$W = \begin{vmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{vmatrix} = -1 - 1 = -2 \neq 0$$

(iii) Find the particular solution by the method of Undetermined Coefficients.

Try $y_p = a_2 t^2 + a_1 t + a_0$

$y_p'' = 2a_2$

insert into D.E.

$$2a_2 - (a_2 t^2 + a_1 t + a_0) = 4t^2$$

like powers of t: $a_2 = -4$; $a_1 = 0$; $2a_2 - a_0 = 0 \Rightarrow a_0 = -8$

$$\boxed{y_p = -4t^2 - 8}$$

Open Book Section

Problem 1. (cont.)
(iv) Find the particular solution by the method of Variation of Parameters.
Hint: work carefully, check your signs and numerical coefficients. The following integral will be useful:

$$\int e^{at}t^2\,dt = (e^{at}t^2)/a - 2e^{at}(at-1)/a^3$$

$$y_p = -y_1\int^t \frac{y_2(4\cancel{g}(s))ds}{W} + y_2\int \frac{y_1 g(s)\,ds}{W}$$

$$= e^t\int^t \frac{e^{-s}(4s^2)\,ds}{(-2)} + e^{-t}\int^t \frac{e^{s}(4s^2)\,ds}{(-2)}$$

$$= 2e^t\int e^{-s}s^2ds - 2e^{-t}\int^t e^s s^2 ds$$

$$= 2e^t\left[\frac{e^{-t}t^2}{(-1)} - \frac{2e^{-t}(-t-1)}{(-1)^3}\right] - 2e^{-t}\left[e^t t^2 - \frac{2e^t(t-1)}{(1)^3}\right]$$

$$= -2t^2 - 4(t+1) - 2t^2 + 4(t-1)$$

$$\boxed{y_p = -4t^2 - 8}$$

Prob. 2. (6 pts.)
Consider the equation for a linear oscillator with frequency = 2:

$$d^2y/dt^2 + 4y = 0: \quad y(0) = 2, \; y'(0) = 4.$$

Express the solution in the form $y = R\cos(2t + \phi)$, i.e. solve this initial value problem and find R and ϕ.

$$y = c_1\cos(2t) + c_2\sin(2t)$$
$$y(0) = 2 \Rightarrow c_1 = 2$$
$$y'(0) = 4 = 2c_2 \Rightarrow c_2 = 2$$
$$y = 2\cos(2t) + 2\sin 2t = R\cos(2t+\phi)$$
$$R^2 = 2^2 + 2^2 = 8 : \boxed{R = 2\sqrt{2}}$$
$$\phi = -\tan^{-1}(1) = \boxed{-\pi/4}$$

Math 256 Midterm #1 *February 7, 2013*

Name: Student number:

Open Book Section

Prob. 3. (10 pts.) (Tank problem!)

A tank of volume V initially containing zero concentration of a solute is supplied with a feed stream at volumetric flow rate Q with $c = c_{in}$. Since there is no production, "accumulation = in – out" and the balance law applied to this problem is

$$V\, dc/dt = Q\,(c_{in} - c(t)): \qquad c(0) = 0.$$

As we have seen, the solution of this initial value problem is $c(t) = c_{in}\,(1 - e^{-(t/\tau)})$, where τ is the characteristic time, $\tau = V/Q$.

Consider the case in which a first order irreversible chemical reaction takes place with rate constant k. The balance law is now

$$V\, dc/dt = Q\,(c_{in} - c(t)) \; - \; k\, c(t): \qquad c(0) = 0.$$

(accumulation = in – out + production (which is negative))

Solve this equation to find the concentration c(t). Use the solution to answer the question: "Does it take a longer or shorter time for the concentration to come to steady state when there is a chemical reaction taking place relative to the case of no reaction?"

$$\frac{dc}{dt} = \frac{Q c_{in}}{V} - \frac{(Q+k)c}{V}$$

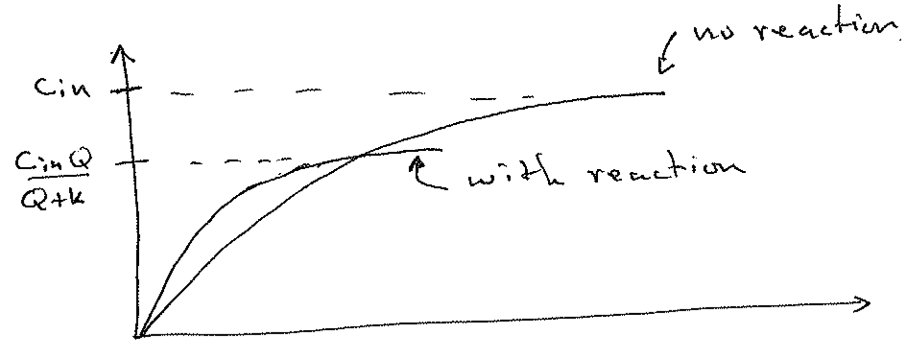
General solution:

$$c = C_1 e^{-\left(\frac{Q+k}{V}\right)t} + \frac{c_{in} Q}{(Q+k)}$$

Apply I.C. to find C_1

$$c(t) = \frac{c_{in} Q}{(Q+k)}\left[1 - e^{-\frac{(Q+k)t}{V}}\right]$$

$e^{-\frac{(Q+k)t}{V}}$ decays faster than $e^{-Qt/V}$



Math 256 Midterm #1 **February 7, 2013**

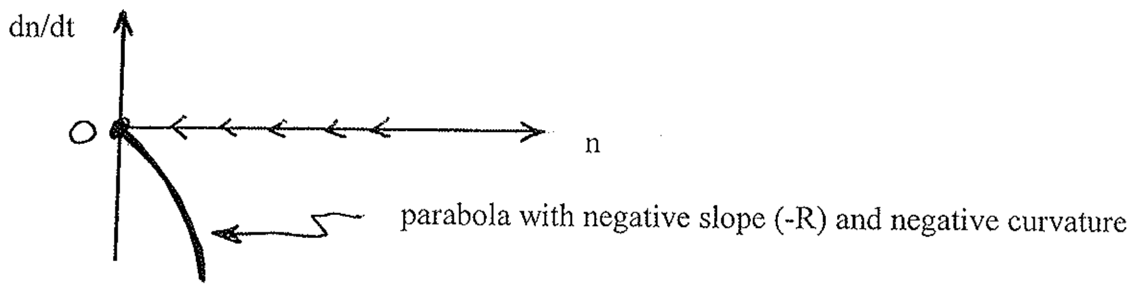
Name: Student number:

Open Book Section

Prob. 4. (8 pts.)
Hockey is a very physical game. Players tend to retire at an early age. They are also vulnerable to concussions. If R is the rate of retirement and C the rate at which players are concussed, a plausible equation for the number of players, n(t), at any given time during the season is

$$dn/dt = -Rn - Cn^2.$$

This equation leads to the qualitative 'phase line' portrait sketched below, indicating that the league will inevitably run out of players, i.e. the only stable steady state is $n = 0$.

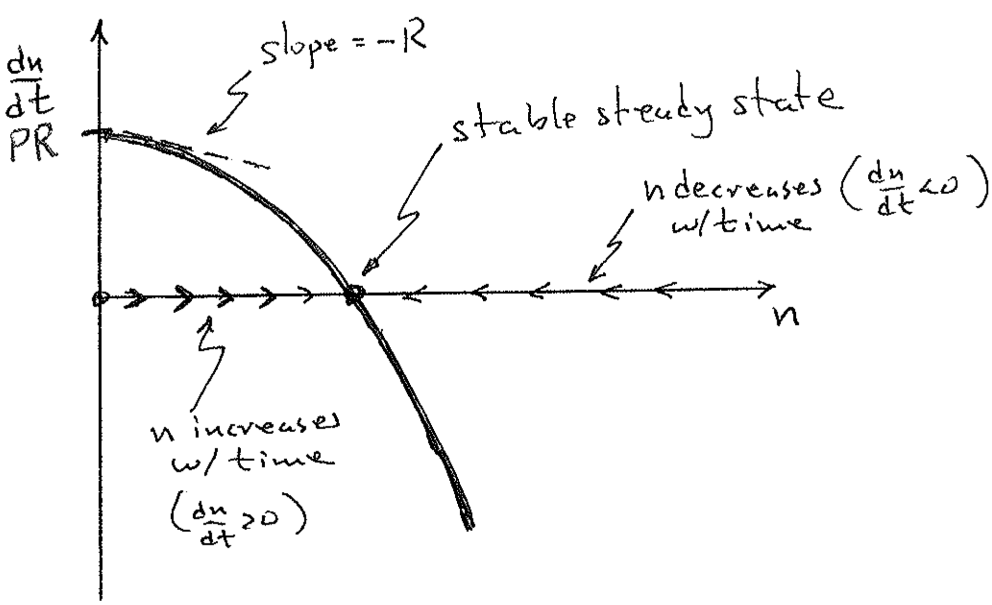


This model ignores the obvious facts that (i) players can be called up during the season and (ii) since there is a semi-infinite number of younger players who would love to play in the NHL, they can be called up at an arbitrarily large rate. Let the rate of player replacement be PR, which is independent of the number of active players and only dependent on the money a team is willing to fork out. Then the modified equation for n(t) becomes:

$$dn/dt = PR - Rn - Cn^2.$$

(i) Analyze the qualitative nature of the solution to this model, using a phase line as above. Show how the phase line illustrates the obvious fact that the NHL can sustain itself through player replacement.

$\frac{dn}{dt} = PR$ when $n = 0$



Math 256 Midterm #1 *February 7, 2013*

Name: Student number:

Closed Book Section

Note: In some of the problems, ' = d/dt

Prob. 1. (2 pts.) Here is a second order differential equation:

$$2\,d^2y/dt^2 - dy/dt = e^t.$$

Does y = exp(t) solve the *non-homogeneous* equation? *Show all your work.*

$y' = e^t \; : \; y'' = e^t \Rightarrow \; 2e^t - e^t = e^t$ ✓

Prob. 2. (3 pts.) Here are three *nonlinear* differential equations. Circle *all* the terms that make them *nonlinear.*

(i) $y'' + t(yy') - (y^2) - t^2 = 0$

(ii) $y' + t(\sin(y)) = 5\,ty$

(iii) $y' + y\sin(t) = (5(ty)^2)$

Prob. 3. (3 pts.) Here are 3 ODEs. *Classify* each one as to whether they are *linear/nonlinear, homogeneous/non-homogeneous, variable/constant coefficient* equations.

	Classification
(i) $y'' + (t-2)\,y' - 4\,y - 5ty = 0$	L, H, VC
(ii) $y'' + y' - 4\,y + t^3 = 0$	L, NH, CC
(iii) $y'' + y^2\,(2 + \sin(t)) - 5y' = 0$	NL, H, VC

Prob. 4. (2 pts.) If L is a *linear operator,* complete the following expressions using the general properties of linear operators:

$L[f(t) + 10\,] =$ $L[f(t)] + L[10]$

$L[10\,f(t)] =$ $10\,L[f(t)]$

Prob. 5. (3 pts.) A harmonic oscillator satisfies the following ODE:

$$d^2y/dt^2 + 4\,y = 0.$$

$r^2 + 4 = 0 \; : \; r = \pm 2i$

Find the two fundamental solutions and demonstrate that they are linearly independent.
Hint: recall that $\sin^2 + \cos^2 = 1$.

$y_1 = \cos(2t) \quad y_2 = \sin(2t)$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos(2t) & \sin(2t) \\ -2\sin(2t) & 2\cos(2t) \end{vmatrix} = 2\left(\cos^2(2t) + \sin^2(2t)\right)$$

$$= 2 \neq 0$$

Math 256 Midterm #1 *February 7, 2013*

Name: Student number:

Closed Book Section

Prob. 6. (4 pts.) A particle is released from rest under the action of a gravitational acceleration, g. The fluid drag on the particle is proportional to both the velocity of the particle and its square. The equation of motion for the particle is of the form:

$$dv/dt = g - a_1 v - a_2 v^2, \quad v(0) = 0.$$

Here a_1 and a_2 are constant drag coefficients.

(i) Is this equation linear or nonlinear? Explain.

Nonlinear because of $a_2 v^2$

(ii) It's first order. Can it be solved using integrating factors? If "YES", explain why (but do not solve). If "NO", then explain how to solve it.

NO - integrating factor only applicable for linear 1st order ODE.

Separable eq:

$$\int \frac{dv}{g - a_1 v - a_2 v^2} = \int dt$$

Prob. 7. (4 pts.) Here is a linear, constant coefficient, homogeneous ODE.

$d^2y/dt^2 + 2\, dy/dt + y = 0$

Do the solutions oscillate in time or not? Do the solutions grow or decay at long times? Explain your answer and show all your work.

Characteristic eq: $r^2 + 2r + 1 = 0$ $\quad r = \frac{-2 \pm \sqrt{4-4}}{2} = -1 \Leftarrow$ double root

$y = c_1 e^{-t} + c_2 t e^{-t} \Rightarrow$ decay at long times